

WHAT IS THE MATHEMATICAL STRUCTURE OF QUANTUM SPACETIME?

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ABSTRACT: *We survey indications from different branches of Physics that the fine scale structure of spacetime is not adequately described by a manifold. Based on the hints we accumulate, we propose a new structure, which we call a quantum topos. In the process of constructing a quantum topos for quantum gravity, we propose a new, operational approach to the problem of the observables in quantum gravity, which leads to a new mathematical point of view on the state sum models.*

I. INTRODUCTION

The problem of finding the quantum theory of gravity has been a fundamental challenge to theoretical Physics for almost a century now. The perspective of this paper is that the mathematical description of spacetime as a smooth manifold inherited from classical general relativity is inappropriate for the quantum theory and needs to be replaced, and that this is the core of the problem.

The discovery of general relativity by Einstein provides an interesting model. Einstein was motivated by considering a rotating wheel within special relativity [8]. Since the circumference of the wheel is along the direction of motion, it is contracted. The radius, on the other hand, is not. One could simply explain this by saying that the wheel is rotating. Einstein, however, was motivated by Mach's principle, and believed that the only difference between a rotating frame and one at rest was the relationship to the distant matter in the universe. He therefore proposed that a gravitational field must have the effect of changing the ratio of the radius of a circle to its circumference.

This one physical idea, motivated by a principle which in the end was not exactly right, turned Einstein's attention to a branch of Mathematics, differential geometry, which had the perfect structure to understand gravity within relativity theory. Within differential geometry, curvature, which can be measured by the deviation of the ratio of the circumference of a circle to its radius from 2π is the central concept.

The adoption of a new mathematical description for spacetime was the critical step in the process of discovery of general relativity. The mathematical simplicity and beauty of Riemannian geometry made the search for the equation of motion and its physical interpretation tractable. In fact, Einstein made a wrong turn at one point and formulated a theory he had to abandon. Nevertheless, the mathematical structure guided him through. Einstein's equation was almost the only possible equation of motion to write which is generally covariant, i.e. well defined in the conceptual language of differential geometry. Differential geometry has a symmetry principle, namely general covariance, which was critical to expressing the principle of relativity.

So, if we are correct that the mathematical structure of a spacetime manifold needs to be replaced, can we imitate Einstein, and find just the physical hint to identify the new structure? Can we find some symmetry principle to guide our search for a theory?

There are several reasons to hope this is possible. Relativists have recently discovered a number of insights into the quantum theory. On the other hand, mathematical approaches to space and geometry which do not rely on an underlying point set are central to a number of areas of pure Mathematics and have made many recent advances. In [10], we made a survey of these mathematical ideas, hopefully accessible to Physicists, and suggested how they could be useful to quantum gravity.

The purpose of this paper is to attempt to construct a specific candidate for the quantum structure of spacetime. In chapter 2, we discuss a number of insights from relativity and quantum theory as to the structure of spacetime, ending in a list of desiderata. In chapter 3, we propose a higher categorical structure, called a quantum topos, which we will define, and whose connections to the physical ideas we explore. In Chapter 4 we will outline construction of a specific quantum topos intimately related to general relativity and to the state sum models [9] for it as well. This part of the program is not finished, but we have an outline.

II. INSIGHTS FROM RELATIVITY AND QUANTUM MECHANICS INTO THE STRUCTURE OF SPACETIME. THE RELATIONAL SETTING

A. Lessons from Quantum Mechanics

There is a fundamental difference between the role of the variables which describe the state of a system in classical and quantum mechanics. In classical mechanics, the variables are regarded as all having objective values, whether we measure them or not. In quantum mechanics, on the contrary, quantities only take on values in the course of a measurement process, which involves an external observer system, and not all variables can attain values at once.

Realist approaches to quantum mechanics have been tried, but they have uniformly failed. The operational approach to quantum mechanics is central to our understanding of it.

What this tells us for quantum general relativity, is that the only meaningful notion of position is apparent position, as viewed from an observer outside the region of spacetime we treat as a quantum process. An observer, treated as classical, which was located inside the region, would cause the geometrodynamical process inside the region to decohere. This motivates the following

DEFINITION *The Relational setting for a quantum theory of gravity is a bounded region of spacetime with a set of experimenters in its causal past who could send probes into it, together with a set of observers in its future who could detect the apparent positions of the probes.*

The quantum theory of gravity must explain correlations and regularities in the results of various experiments, that is to say, the apparent positions of different probes as seen by different observers in the relational setting.

In order to interpret the theory, we need to assume that the particular quantum state of our spacetime is reproducible, so that we can think of ensembles of experiments and study relative probabilities.

A very important point in the interpretation of quantum mechanics is that the act of measurement disturbs the system. In the context of quantum gravity, that means that it is not possible to think of experiments involving idealised probes that do not react on the spacetime geometry. Theory must explain only real experiments, in which forming a black hole is an extreme possibility.

An important aspect of the interpretation of quantum mechanics is that “world elements”, or propositions about a system which are either true or false, do not form a distributive lattice (see appendix B). This contrasts to the subsets (open or measurable) of a manifold.

We are led to consider the possibility that the relational setting will lead to a nondistributive lattice of observable subregions in quantum spacetime.

B. Relative observation and curvature in general relativity

The fundamental equation of general relativity relates the curvature of the spacetime manifold to the distribution of matter. The curvature of a manifold can be understood as the rate at which nearby geodesics deviate. Thus, it is possible to understand classical general relativity as a set of laws which determine where the same event will appear to be to different observers. Naively, one might think that an observer would interpret the apparent positions of events as located in the causal past of a copy of Minkowski space whose zero point coincides with the observer. In fact, however, the curvature of the spacetime can cause a shear in the congruence of null geodesics radiating from an event. This means that apparent positions of past events occupy points in a circle bundle over the past light cone of the observer. This fact, which seems to have gone unremarked, plays a very important role in the quantum theory of apparent position, as we shall discuss below.

Neglecting this effect, by observing correlations between where particular events appear to be to several observers, an experimenter could identify the causal pasts of different observers and reproduce the spacetime manifold, or at any rate the part of it the observers could see. This is an operational version of the standard definition of a manifold in terms of charts and transition functions. The observers in such a process need to have two eyes, or a wide field of vision, in order to detect parallax information.

Thus, the data of classical general relativity would appear in the relational setting as correlations between where different observers see the same probe. It would be an interesting and useful exercise to formulate general relativity explicitly in such a form.

Now let us try to imagine how the correlations between apparent positions for different observers might change as we go from classical to quantum general

relativity. Imagine for a moment a quantum state of the metric for the region which was a superposition of two classical spacetime metrics. A region in one metric would not appear to be consistently within any region of the other metric of the same size to all observers, if the region were small compared to the overall curvature. On the other hand, it would appear to be inside a larger region in the second metric to any observer. This leads directly to a failure of the distributive law for observable regions, consistent with the behavior of “world elements” in quantum mechanics (see Appendix B).

C. The Planck scale

The oldest indication from Physics that the continuum of classical Mathematics might not apply to physical spacetime is the existence of the Planck scale. This length, which is far too small to be practically observable, is the result of the interaction of general relativity with quantum mechanics.

To recapitulate for the mathematical reader, quantum mechanics tells us that the uncertainties of position and momentum of any body are inversely related. In order to confine it to a small space, it must have a high probability of having a large momentum, and thus a large energy.

General relativity, on the other hand, tells us that any body deforms the causal structure of the spacetime around it. This is closely related to the red shift near any massive body. If the energy density in a small region is sufficiently high, it creates a black hole around itself, so that no information from it can ever escape to the distant universe. This can be thought of as an infinite red shift.

The combination of these two effects means that no length less than a certain scale, called the Planck length or l_p can ever be observed, because any probe concentrated in so small a region would contain so much energy that it disappeared into a black hole. This distance is approximately 10^{-33} centimeter, much too small to be detected directly.

In a classical theory this plays a minor role. However if we insist on an operational approach to quantum geometry, it points to a very profound departure from the classical continuum point set.

In the context of the relational setting, this places serious limits on the amount of information about the geometry which can be observed. More recent developments within classical and semiclassical general relativity extend this quite considerably, as we shall discuss.

D. Lessons from quantum field theory. Dimensional regularization.

The development of quantum field theory provides similar suggestions that the classical spacetime continuum needs replacing. In the first place, there is the problem of the ultraviolet divergences. Terms in the Feynman perturbation series can be indexed by Feynman diagrams. To compute the actual contribution of a given diagram to an amplitude, we must take the integral over all ways to embed the diagram into spacetime, inserting propagators related to the

spacetime geometry on the edges. These integrals are not in general finite. The important divergences are concentrated in the region of the multiple integral where one or more loops shrinks to zero length.

This problem is solved by a complicated subtraction procedure called renormalization. The solution has no real theoretical motivation; different renormalizations can give different answers to the same problem.

If we try to treat gravity as a quantum field theory, the subtraction procedure breaks down. It is a “nonrenormalizable” theory.

The fact that all theoretical progress in particle Physics for the last half century has depended on finding tricks that have no conceptual justification has colored the entire field. I think it is not duly recognized that such successes must be taken as provisional.

Renormalization generally begins by cutting off the distance scales at some minimal length, for example by doing the theory on a discrete lattice.

This seems to suggest that the short distance structure of spacetime is very different from a classical continuum.

The most successful theories in particle physics are the nonabelian gauge theories. The development began with the discovery by t’Hooft and Veltman [11] that gauge theories can be renormalized via dimensional regularization. Before the discovery of dimensional regularization, nonabelian gauge theory was believed to be nonrenormalizable.

In dimensional regularization, spacetime is treated as if it had $4 - \epsilon$ dimensions. This is accomplished by analytically continuing the evaluations of integrals as a function of the spacetime dimension. There is no mathematical understanding of the meaning of this procedure, despite its central role in our understanding of nature.

We need to be wary of the common misunderstanding that dimensional regularization (or renormalisation in general) is a computational device. The quantities to be calculated are mathematically undefined until the renormalisation procedure is chosen, so it forms part of the hypothetical structure of the theory. As such the meaning of dimensional regularisation is not understood. If it is not a hypothesis about the fine structure of spacetime, I have no idea what it could be.

One could try to interpret this procedure by saying that at very short scales, the number of observable regions of a given size was different from what one would expect by counting subsets of a continuum.

E. Black hole thermodynamics and holography

A series of discoveries about the classical and semiclassical behavior of black holes has led to the idea that a black hole should be regarded as having a finite entropy, equal to the area of its boundary measured in Planck units.

$$E = A/(l_p)^2$$

This is supported by results which describe collisions of black holes, matter falling into black holes, and most strikingly, by black hole radiation.

Since no information from the interior of a black hole can escape, it is natural to interpret this as due to the state of the quantum fields in the region of the skin of the black hole. A naive counting of all modes in the quantum fields would give an infinite dimensional Hilbert space, and therefore an infinite entropy. In order to give a statistical explanation of the entropy (= logarithm of the number of microstates) it is necessary to assume the quantum field is cut off at the energy corresponding to the Planck length.

Now a black hole is a global phenomenon, nothing remarkable happens near its boundary locally. So if we assume that QFT is cut off there, it should be cut off everywhere. This is further indication of the nonapplicability of the classical continuum.

The ideas about black hole thermodynamics have implications for arbitrary bounded regions of spacetime. In a body of work which goes by the name of holography, it has been shown that any bounded region of a suitably causal spacetime can only pass to its environment information from a finite dimensional Hilbert space, with dimension given by the entropy of a black hole which would replace the region, i.e. its area in Planck units (whence the analogy to a hologram.)

This has immediate implications for the relational setting we have been proposing. Only finitely many probes can be admitted without turning the region into a black hole, which emits only thermal noise. This means the description of the relational setting will be cut off, and contain only finite information for any given region.

F. State sum models

Recently a model [9] (more precisely, a closely connected set of models) has been proposed for quantum general relativity in four dimensions. These models are associated with a four dimensional simplicial complex, which can be thought of as a triangulation of a spacetime manifold, but does not need to be.

In order to construct the models, we need to make use of the unitary representations of the Lorentz group, as defined by Gelfand and Harish-Chandra. (In some variants, we use the representations of the Quantum Lorentz Algebra instead). These representations denoted $R(k, \rho)$, are indexed by one real and one integer parameter, which can be thought of as forming a complex spin $1/2k + i\rho$, since the lorentz algebra is a complexification of the ordinary 3D rotation algebra.

In constructing the model, we used only intertwining operators between representations; in other words, everything we did was manifestly Lorentz invariant. This has the possibility of playing the role for quantum gravity that general covariance did for classical general relativity.

The model is constructed by assigning irreducible representations to the 2-faces of a triangulation of the spacetime, putting in a special intertwiner across each tetrahedron, tracing around each four simplex, multiplying over four simplices, then summing over all labelings [9].

The representations are quantizations of the directed area elements, or bivectors, which describe the areas and orientations of the faces.

In order to be a quantization of the bivectors on triangles only, the irreducible representations must satisfy a constraint; they must be balanced. In the above notation, $k=0$. This causes the resulting state sum to be nontopological, because the balanced representations do not close under tensor product. We shall see below that this recurs as a central question in the quantization of apparent positions as well.

A significant number of the fundamental geometrical variables in Minkowski spacetime, such as areas volumes and dihedral angles, have natural quantizations as operators on these representations and their tensor products. The geometry of a composite system (several faces, for example), can be quantized on the tensor product of the quantizations of the pieces.

In studying the state sum models we discovered that not all the operators corresponding to geometric variables commute with one another. For example, the operators giving the two shape parameters of a tetrahedron, which can be thought of as two adjacent dihedral angles, do not commute.

The state sum models give us several suggestions about how regions in quantum spacetime should be described:

1. Unitary representations of the Lorentz algebra should appear as building blocks, and all maps between them should intertwine Lorentz symmetry.
2. Regions should be modelled as tensor categories, in which parts are tensored to make a whole. This is also suggested by standard quantum mechanics as applied to compound systems, such as multiple spins.
3. The state sum models themselves should appear as approximations to the full quantum theory of gravity.
4. The geometrical variables will not all commute.

SUMMARY

We have made the observation that four important areas of fundamental Physics, renormalizable quantum field theory, gauge theory, classical general relativity in combination with quantum mechanics, and the theory of semiclassical black holes, all point to the conclusion that the classical continuum model of spacetime must be abandoned.

Furthermore the clues which lead us to this conclusion are in every case central to the field of Physics from which they come. They are steps without clear justification, without which the theory is unable to make contact with experiment.

It is a bit mysterious why these ideas have not been connected more prominently, or taken as indicating an important direction. Perhaps the chasm which

has opened between Mathematics and Physics in our educational system has something to do with it.

We have now found a number of physical indications of what the quantum theory of spacetime should look like. We collect these for future reference, connected to some mathematical ideas they seem to suggest.

list of desiderata for quantum spacetime

α : *minimum length scale- no point set continuum*

β : *relational geometry-sheaves over a site of observers*

γ : *finite information transfer- red shift as filter*

δ : *non-commutative variables-operational interpretation*

ϵ : *Lorentz invariance-Gelfand representations as building blocks, and Lorentz covariant functions between them.*

ζ : *The density of regions of a given size should follow a different power law from that naively predicted by a spacetime manifold*

ι : *The regions of spacetime should not obey the distributive law.*

In the following sections, we shall show how a mathematical construction called a quantum topos naturally accommodates these criteria.

III. QUANTALOIDS AND THEIR SHEAVES

A. Quantaloids for Physicists.

Before giving a mathematical introduction to the type of mathematical structure I am proposing for quantum spacetime, I want to explain what they are and how they contain what the theory should contain in simple physical terms.

Since we cannot detect points, the geometry we can observe consists of regions. Regions can contain one another, we can form unions and intersections of them, and these operations satisfy obvious algebraic laws.

Such structures are called lattices (no relation of the kind quantum field theory is sometimes done on) see appendix A. We can construct a lattice by starting with a set of points, and specifying certain subsets to be members of the lattice. Not every lattice can be constructed in this way, however, and it is not necessary to assume that the physically observable regions have a point set interpretation.

So we could begin by replacing the spacetime manifold with a lattice. This would be a theory somewhat similar to causal sets, in that the discrete point set in the causal set picture has a partial order on it.

However, this would be an “absolute” spacetime, not a relational one. So a better approximation would be to describe the structure of a spacetime as a lattice of observable regions for each observer. There would then need to be some consistency relations between what different observers see, which would be expressed by “lattice maps”.

So we need a structure of many lattices with maps of lattices between them to construct a relational spacetime. This is what is called a quantaloid. The “quant” in the name refers to quantum mechanics.

It is interesting that the history of the theory of quantaloids is a convergence of two very different area of Mathematics. One is a natural but abstract direction in higher category theory, the other is an attempt to interpret quantum mechanical systems as noncommutative geometries.

B. Quantaloids, definitions and examples.

DEFINITION: *A quantaloid is a category enriched in the category of sup-complete lattices.*

See appendix A for the definition and basic properties of a sup complete lattice.

Let us unpack this for the non-categorical reader. There is a set of objects, called the objects of the quantaloid. Between any two objects A and B there is a sup complete lattice called $\text{Hom}(A, B)$, and for any three objects A, B, C, of the quantaloid there is a tensor morphism of sup complete lattices

$$C_{A, B, C} : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

called composition.

In this definition, \times denotes the tensor product of sup complete lattices.

The operation of composition must distribute over arbitrary sups, i.e.:

$$a \circ (\vee b_i) = \vee (a \circ b_i);$$

this is a consequence of the definition of an enriched category which we shall omit here.

Examples of quantaloids

1. The free quantaloid on a category.

Given any category C we can define a quantaloid $P(C)$ whose objects are the objects of C. If A and B are objects of C, then the lattice $\text{Hom}_{P(C)}(A, B)$ is the lattice of subsets of $\text{Hom}_C(A, B)$. The composition is defined as

$$U \circ V = \{uv \mid u \in U; v \in V\}.$$

So we can see that any mathematical structure has a quantaloid version, where we focus on subsets instead of elements.

2. Quantales

DEFINITION: *a quantaloid with one object is a quantale.*

Unpacking, this means that we have a sup complete lattice (the maps from the object to itself) with a multiplication which distributes over joins. Any semigroup gives us a quantale, namely the lattice of its subsets with

$$A \circ B = \{ab : a \in A, b \in B\}.$$

Algebraic structures richer than semigroups have quantalic versions which reflect their additional operations. For any vector space we can construct the quantale whose objects are subspaces and whose product is given by

$$A \circ B = \{\sum x_i a_i b_i : x_i \in C, a_i \in A, b_i \in B\} ;$$

where C is the complex numbers.

This is a subquantale of the quantale of all subsets of the vector space under $+$, and can be constructed by operating on that quantale with the operation of taking the linear closure. This is an example of a quantalic nucleus, as we shall explain below.

Rings, algebras, and especially C^* -algebras provide physically interesting examples of quantales. The subsets of a ring form a quantale under multiplication. The additive subgroups form a subquantale, given as the image of the first under the operation of additive closure, another quantalic nucleus. Left, right and 2-sided ideals also form quantales. Similarly, the subspaces and ideals of an algebra form quantales, where we have closed under the operation of forming the linear span.

An especially interesting example for Physics is the quantale of closed right ideals $Q_R(A)$, where A is a C^* -algebra. The objects of this quantale are the closed right ideals of A, and

$$B \circ D = CL\{\sum x_i b_i d_i c_i \in C, d_i \in D, b_i \in B\} ;$$

where CL means topological closure, x_i are complex numbers, and B and D are closed right ideals of A.

In the case of a commutative C^* -algebra, this reduces to the spectrum of the algebra, considered as a locale. Every quantum mechanical system has a quantale associated with its C^* -algebra of observables, which can be thought of as a non-commutative geometric structure.

3. Rel(C)

To any category C we can associate the quantaloid $\text{Rel}(C)$ of relations. The category of relations of the category of sets has sets for objects and the lattice of subsets of $A \times B$ as $\text{Hom}(A, B)$. This is equivalent to substituting relations for functions.

To generalize this to an arbitrary category C we first construct the category of spans of C . The objects are just the objects of C , while a span between two objects A and B of C is a diagram:

$$B \leftarrow D \rightarrow A$$

where D is any object of C .

Intuitively, we can imagine the graph of this diagram as a “subset “ of $B \times C$. Now to solve the problem that different spans might have the same graph, we make a rather technical definition

Definition: a crible is a set of spans which whenever it contains

$$B \leftarrow D \rightarrow A$$

also contains the span generated by composing with any map $E \rightarrow D$.

Cribles correspond to subsets of the cartesian product in the category of sets. In a general category, the cribles between two objects form a sup complete lattice, and in general form a quantaloid.

C. Quantaloidal nuclei and Grothendieck topologies.

We have a very general method of forming new quantaloids from old ones by forming a quotient quantaloid. This involves the concept of a quantaloidal nucleus.

Definition : Let Q be a quantaloid. A quantaloidal nucleus on Q is an assignment of a map $j_{a,b}$ for each pair of objects a, b of Q such that:

1. $f \leq j(f)$
2. $j^2 = j$
3. $j_{a,b}(f) \circ j_{b,c}(g) \leq j_{a,c}(f \circ g)$

Now it is a well known theorem that quantaloidal nuclei correspond 1-1 to quotient quantaloids. The image of any j satisfying 1-3 is again a quantaloid, and any quotient quantaloid is so obtained. Furthermore, any quantaloid is a quotient of a free quantaloid (example 1).

Quantaloidal nuclei are a generalisation of closure operators. The quantaloidal nucleus which assigns to any subset of a vector space its linear span

gives the quantale of linear subspaces as a quotient of the quantale of subsets. Similarly, the quantale of right ideals of an algebra can be obtained from the quantale of its linear subspaces by letting j give the ideal spanned by the subspace.

If we take any category C , the quantaloid $\text{Rel}(C)$ has the property that Grothendieck topologies on the site of C correspond to quantalic nuclei on $\text{Rel}(C)$ which respect intersection. In fact quantaloids were invented, although not so named in [6] precisely because they are an easier to work with approach to Grothendieck topologies.

Any topos can be constructed as sheaves over some quantaloid. The quantaloids that so appear are like many-object versions of locales, they have commutative multiplication and distributive lattice structure. So we see that the suggestion below that quantum gravity lives in the sheaves over a quantaloid is a direct generalisation of the suggestion that it lives in a topos, and can be thought of as a noncommutative version, or a quantization of the older hypothesis.

D. Quantum topoi and geometry.

Now there is a natural notion of presheaves over a quantaloid. It consists of a set over each object of the quantaloid, and a member of the lattice $L_{x,y} \in \text{Hom}(A, B)$ for each pair of one element x in the set over A and one y in the set over B ; satisfying the consistency relation:

$$L_{x,y} \times L_{y,z} \leq L_{x,z}$$

which the categorically minded will recognize as the definition of a weak or lax functor.

Sheaves are defined by the usual glueing property on all open covers.

In the quantaloidal picture there is a unique topology associated to the quantaloid. We would pass to the analog of a new Grothendieck topology (appendix A) by passing to a quotient quantaloid.

Furthermore, the category of sheaves over a quantaloid is equivalent to the category of presheaves [4], so the theory of quantaloidal nuclei has fully absorbed the subtleties of Grothendieck topology.

Let us make the following

Definition: the category of sheaves over a quantaloid is a quantum topos.

The sheaves over a quantaloid themselves form a quantaloid.

If the quantaloid is a locale, we reproduce the ordinary definition of a localic topos.

To repeat, our suggested definition of a quantum topos extends the concept of a topos from commutative to noncommutative geometry.

Now quantum topoi can be viewed in many ways. They are equivalent to categories enriched over the base quantaloid, and to variable sets, or sets with relative equality relations.

One of the interesting ways of thinking of quantum topoi is that they represent a kind of geometry. The example due to Lawvere [7] illustrates this. We define the locale $R^{+\leq}$ as the lattice of sets of the form $\{x : 0 \leq x \leq a\}$ for some positive a .

Presheaves over this locale are exactly metric spaces. The lax condition mentioned above gives the triangle inequality. Sheaves are cauchy complete metric spaces, and the equivalence of the two categories comes about via cauchy extension of functions.

If we think of these presheaves as sets with variable equality we get the picture that two points in a metric space distance r apart are equal up to stage r . This is an interesting precursor to a relational geometry in which observers receive only finite information.

Now the quantale in the above example is a locale, so commutative and distributive. It is not surprising the geometry it generates is classical. Passing to a suitable noncommutative quantale is a natural road to a quantum geometry.

Another important aspect of topos theory is that by doing Mathematics in a topos we adopt its logical structure, which in the case of a localic topos comes about by thinking of the underlying locale as a complete Heyting algebra [12].

Now the original motivation for this work was the idea of Isham that Physics should be done in a topos which would lie over a locale representing all classical worlds, or perhaps classical states of an observer.

However, in a later paper [13], Isham shows that the internal logic of a topos can give us only intuitionistic logic, not quantum logic; and that it is therefore necessary to revert to a realist interpretation of quantum mechanics.

This is a profound weakness in a promising new direction. Passing to quantum topoi resolves this, since the internal logic of a quantaloid is quantum logic. this is technically difficult to show, but no real surprise, considering that we have a nondistributive lattice with a noncommutative multiplication at the base.

Categorically minded readers will no doubt know that Grothendieck topoi have two definitions, one as above i. e. sheaves over a site, the other axiomatic. We do not know of an intrinsic characterization of what we have called quantum topoi, but would like to have one.

E. Summary of properties of quantaloids

Before we go on to try to create a physical theory; let us think about how our list of desiderata above correspond to the structure of quantaloids and quantum topoi in general.

Quantaloids have a very rich family of quotient spaces, making it easy to filter out information. This makes us optimistic about desiderata α, γ, ζ . We are already in a category of sheaves, which helps us with β , and the algebraic structure of a quantaloid has noncommutative multiplication and a nondistributive lattice, dovetailing with δ and ζ . So a priori, it seems that we have a useful mathematical setting. The fact that it is a far reaching generalization of the theory of metric spaces as well is also suggestive.

Finally, we would expect the quantum theory of gravity to resemble quantum theory as we know it. Since each quantum mechanical system already can be described as a quantale, we have a starting point which is physically familiar, except for the abstract language, which the author hopes the physical reader will eventually learn to love.

IV. BUILDING A QUANTUM TOPOS FROM RELATIVISTIC OBSERVERS

A. Physical overview

Now we want to construct a quantum topos which would be an appropriate setting for quantum gravity.

First let us state the problem physically. We believe that an operational interpretation of quantum mechanics means that only positions of regions as they appear to observers, and correlations of apparent positions, can appear in the theory. Observable regions for each observer form a lattice. To form the kinematics of our theory we must find some way to combine the lattices. The dynamics of the theory then must constrain the possible correlations, and their time evolution. A reasonable approach would be to search for a quantaloidal nucleus to implement the dynamics.

B. General Program

The mathematical program is the following:

1. Make a quantum mechanical model of an observer in general relativity.
2. Construct the quantale corresponding to the observer.
3. Construct the quantaloid nucleus corresponding to the red shift relating the observer to a region.
4. Form the quantaloid of all observers for the relational setting.
5. Study the quantum topos of sheaves over the category of observers. As explained above, it is itself a quantaloid.
6. Impose Einstein's equation as a quantaloidal nucleus on the quantum topos.

Now we have not yet completed this program. We shall carry it out as far as we know how, then give an outline of the remaining steps. We discuss below the possibility that some further shift in point of view will be necessary.

C. Observers. One eye and two.

An observer at a point in a spacetime observes incoming information about the location of past events on a 2-sphere of null lines. This 2-sphere has an action of the Lorentz group which coincides with the action of the group of fractional linear complex transformations on the Riemann sphere $CP^1 = S^2$. In other words, the 2-sphere of null lines inherits a complex structure from the action of the Lorentz group on Minkowski space.

A single observer who only marked the apparent position of an event on a copy of CP^1 would not be able to observe the distance of the event, and would not be able to infer its time either. To produce a mathematical description of an observer who could make such a determination, we could either combine two nearby observers with two nearby copies of CP^1 ; or else think of the observer as nontrivially wide, and keep track of rays from a common event which impinged on different points of the 2-sphere at slightly different angles, which did not converge at the center. The second possibility is perhaps more physical, but we choose the first in order to construct a mathematically idealised observer which would be easier to quantize.

So for us an observer has “two eyes,” by which we mean two nearby copies of a CP^1 of null lines, with a parallelism defined between them, from which the apparent position of a distant event could be defined in an ideal past in Minkowski space via parallax.

We then want to have a family of such observers, with relative positions and orientations specified, and to keep track of the correlations between their observations of events in some past region, in order to reproduce the geometrodynamics of some spacetime region in their common past. Classically, we could organize our observations as subsets of the cartesian product of the CP^1 s where correlations appeared. The results would be somewhat complicated even classically by the presence of gravitational lensing and consequent multiple images. regions in event horizons would not appear unless we had observers inside them.

Now again classically, we could embed such a description in a quantaloid. The objects would correspond to the observers, and the hom lattices would be the lattices of all subsets of the cartesian products of the corresponding CP^1 s, a construction we referred to as the category of relations above, over the category whose objects are the observers and whose morphisms are functions on the spheres. Let us refer to this quantaloid as the (classical) Relational Observation Quantaloid..

If we wanted to include limitations on the accuracy of distance measurements in a description of classical relativity, we could do so by constructing a quantaloidal nucleus on the ROQ. (This is a philosophically unattractive construction, since in classical theories the disturbance of the system by measurements is ignored, and since the Planck scale is a quantum effect, but it is worth thinking about as a comprehensible toy problem).

We would then use a structure analogous to the construction of Lawvere mentioned above to construct a quantaloid of successively fuzzy geometries on the copies of CP^1 and their products.

At this point we must make an important observation. In general relativity, the Weil tensor can induce a shear on a congruence of null geodesics. This means that, to a two eyed observer, a point in the past does not necessarily appear as a pair of null lines which form a plane together with the segment connecting the centers of the two eyes. The set of all possible apparent past events would appear as a circle bundle over the past light cone, where the circular parameter would be given by the dihedral angle between the two planes determined by the two null rays. This can also be described as the cartesian product of the two

2-spheres, with the diagonal removed, since diagonal points would appear at infinity. We are using the assumed local parallelism between the two 2-spheres, which can be thought of as parallel propagation along the segment joining their focal events.

Now if we want to pass to a theory based on quantum mechanical observers, we need to study the Hilbert space $L^2(CP^1)$, together with tensor products of copies of it with itself to represent observers. We would also need to keep track of the action of the Lorentz group on these Hilbert spaces to compare observers in moving frames. In short, we need the structure of the Hilbert spaces of observers as representations of the Lorentz group; in particular in order to model the relationships between moving observers.

This brings us directly to consider the Gelfand representations.

As we mentioned in the section on state sum models, the mathematical structure of the unitary representations of the Lorentz group is a quantum geometry of Minkowski space. It is not surprising that it gives us a description of an ideal quantized observer in general relativity.

In his study of the unitary representations [1], Gelfand studied the Hilbert spaces of various homogeneous spaces for the Lorentz group, and related them by means of integral transforms, to obtain the representations and study their behavior under tensor product.

Three important homogeneous spaces for our purpose are the CP^1 of null lines through an event, the past null cone of a point NC, and the complex plane C^2 . The last gets its action from the isomorphism of the Lorentz group with $SL(2, C)$. Gelfand shows that there is an invertible integral transform between $CP^2 \otimes CP^2$ and C^2 , where the restricted product discussed above is meant.

This means that as representations of the Lorentz group under the natural actions

$$L^2(CP^2 \otimes CP^2) \cong L^2(C^2)$$

Now the decomposition of these function spaces as irreducible representations is known. A single CP^2 gives a single representation $R(0,0)$. It can give any nonzero ρ if we modify the action of the group to include a power of the Jacobian. The Hilbert space of the past cone contains all the Gelfand representations with zero k and arbitrary ρ .

$$L^2(NC) \cong \int R(0, \rho) d\rho$$

which is a direct integral of representations in the sense of Mackey [15].

Now Gelfand's integral transform tells us that the Hilbert space on the product of two copies of CP^1 , the Hilbert space of a two eyed observer is equal to the Hilbert space on C^2 , which is the sum over k of the integral above:

$$L^2(CP^1 \otimes CP^1) \cong \sum_k \int R(k, \rho) d\rho.$$

We now take this as a construction of a Hilbert space for a two eyed observer with perfect vision. We denote it H^B_i for the binocular Hilbert space of the i th observer.

The combination of Gelfand representations which appear on the cone is precisely the subset of balanced representations which appear in the BC model [9]. The set which appear over C^2 is a full tensor category. If the BC state sum model were extended to include all those representations, it would become a topological field theory.

The observation that a classical two eyed observer would see points as lying in a circle bundle over spacetime translates in the quantum version into saying that the constraints in the BC model are partially relaxed for distant observers. We shall use this idea below to make a conjecture as to the dynamics of quantum gravity.

D. Redshifts and projections. Kinematics

In a physical example of the relational setting, there would have to be a projection on H^B_i corresponding to the information which could flow from the observed region to the observer. This is closely related to the phenomenon of the red shift in general relativity. As we approach the event horizon of a black hole, the red shift tends to infinity, so the flow of information outward goes to zero. The finite information theorems cited above are consequences of this.

Since any observer must have a mass bounded below by the uncertainty principle and above by its Schwarzschild mass, real observers with two eyes would see quotient spaces of the ideal observer constructed above.

This part of our program has not been completed. The quotient Hilbert spaces of mutually moving observers will not completely overlap, since rest energies in their frames will differ.

Assuming we are able to formulate this, the space of projections on the physical Hilbert space of each observer will form a quantale. Extending this by the category of relations construction will form the quantaloid of observation of any ensemble of physical observers, whose objects will be any set of the physical observers, to which will correspond the lattice of all correlated observed positions observed by the set of observers, combined by taking the tensor product of lattices.

DEFINITION: *The physical quantum topos is the topos of sheaves over the quantaloid of observation in the above paragraph.*

This is a mathematically useful way of describing all lattices of simultaneous observed positions.

E. Dynamics, a conjecture

So to formulate a theory, we need some way of computing the probability that some probe will appear simultaneously in some apparent regions to some

set of real observers. We can describe probes dually as future light cones in the past of the observed region. They would also have projections on their dual or time reversed Hilbert spaces, due to the theorems on finite information transfer.

The mathematical similarity between the ideal observers and the construction of the BC model is not surprising. A two eyed observer is essentially observing a long thin bivector represented by a triangle. The BC model used the bivectors of a triangulation as basic variables [9]. We can interpret the constraint of the BC model as saying that when the nearby observers on its vertices observe on another the images they see are not unfocussed by a shear because the space between them is locally flat.

This leads to the following :

CONJECTURE: *The relational probabilities for the apparent positions of a probe in a physical quantum topos can be calculated by tensoring the state of a probe into the appropriate site of a BC model, then tensoring the resultant representations through the rest of the triangulation, and measuring the probability amplitudes on the future boundary. A single triangulation will suffice if it is fine enough to contain all of the physical information which the real observers can see.*

Any probe will have the effect of introducing unbalanced representations into the state sum. This will drive it toward being a topological field theory. A sufficiently strong probe, or a sufficiently large number of them will make the region appear topological. The physical sign of this will be that outside observers will see an uncorrelated thermal flux, with no information about the interior geometry. The tqft state will therefore look exactly like a black hole. This is consistent with the work in relativity which models the horizon of a black hole with the CSW tqft. The relational version of a black hole is a topological state.

This conjecture is at a very preliminary stage. We state it despite its vague formulation because it is at the nexus of a beautiful combination of ideas. In particular, the convergence of the analysis of observation in general relativity with the mathematical foundation of the state sum models seems compelling, at least to its possibly doting father.

F. Summary

To say what we have proposed in nonmathematical language, it may be helpful to compare a small region of spacetime to the appearance of a chamber through a small hole in a thick wall at a finite temperature. The region inside seems fuzzy, because the thermal state of photons in equilibrium with the walls obscures the details.

We would not be tempted to think that the fuzzy apparent geometry was real, because we could always remove the objects inside the chamber and measure their shapes.

In the case of a small region, we cannot dissect it or enter it. The fuzzy observed geometry is all we ever detect. If we think of virtual processes in which the small region interacts with the exterior, the fine details of some hypothetical unobservable interior geometry could not be communicated to the exterior. Feynman diagrams should properly only be integrated over the fuzzy geometry.

A mathematical description of fuzzy geometry is a difficult matter. It is not really a finite point set. rather it is a complicated lattice of minimal observable regions. Quantaloids are the natural language to study such geometry.

G. Further directions. Not categorical enough?

The foundation of the work up to this point is the notion of an observed region. We are asking for probability amplitudes for where some material probe appears to be to different observers. In making this the point of departure, we are ignoring the specific physical character of the probe.

It is not entirely clear that we are justified in doing this. Quantum gravity only becomes significant at energies where large neutral probes would become unstable and likely to decompose. Highly curved regions of spacetime would produce strong excitations of the local matter fields; turning them off may be completely unphysical.

So we may have to refine our approach to ask for regions where an elementary particle of a given type may appear; the excitations of gauge fields from the standard model may mean that the correlations for different types of probes with different charges are different.

Put differently, the quantum theory of gravity may well not exist except as a sector of a geometric unified theory. The geometric interpretation of gauge theory suggests as much.

Now it is interesting that still another branch of Mathematics exists, which could provide a natural setting for such an approach. I am referring to the theory of Grothendieck categories (same Grothendieck, different categories).

In this body of work, A space is represented by the tensor category of coherent or quasicohherent sheaves over it. This category is axiomatized, and abstract Grothendieck categories are treated intrinsically as topological structures, with the role of regions played by certain subcategories [14].

Without going into the technical details, coherent sheaves are a generalization of bundles; so the particle physics in a region of spacetime could be thought of as a Grothendieck category, in which the bundle specifies the particle type, while the tensor product contains the interactions, thought of as Feynman vertices.

The theory of Grothendieck categories has turned out to be a powerful mathematical tool; it is the foundation of the theory of noncommutative scheme theory, i.e. the effort to understand algebraic geometry with noncommuting variables.

There is an important part of semiclassical general relativity which has not fitted in a natural way into our quantum topos picture. We refer to the Unruh

and Hawking radiation effects. Both of these link the spacetime geometry to a thermal state of the matter fields.

It is much easier to include this in a Grothendieck categorical approach, where the matter fields are included into the fundamental spacetime structure.

Perhaps a quantum Grothendieck topos with a thermal functor will emerge as the final setting.

As I have learned more about the categorical approaches to pointless topology and geometry, it has seemed to me that there was massive parallelism with the interesting issues in quantum Physics and relativity. The connection between Grothendieck categories and Feynman's approach to quantum field theory is a striking example of this.

APPENDIX A. Sup complete lattices, locales presheaves and topoi.

DEFINITION: *A partially ordered set is a set equipped with a relation \leq such that*

1. $a \leq b$
2. if $a \leq b$ and $b \leq c$ then $a \leq c$
3. if $a \leq b$ and $b \leq a$, then $a=b$.

DEFINITION: *A lattice is a partially ordered set with (finite) sups and infs. That is for any two elements a and b , and therefore for any finite set of elements, there exist elements $a \wedge b$ and $a \vee b$ such that any element less than both a and b is less than $a \wedge b$ and dually for $a \vee b$.*

DEFINITION: *A sup complete lattice is a lattice with infinite sups. That is, for any collection I , finite or infinite, of elements of the lattice, there exists an element \vee_I such that any element greater than all the members of I is also greater than \vee_I*

It is a simple theorem that a sup complete lattice also has infinite infs.

DEFINITION: *A frame or locale is a sup complete lattice which satisfies the distributive law*

$$a \wedge (\vee_{i \in I} b_i) = \vee_{i \in I} (a \wedge b_i)$$

The most obvious way to obtain a sup complete lattice is to take the set of subsets of a set with the obvious set theoretical operations.

Given sup-complete lattices L, M ; the hom set $\text{Hom}(L, M)$ is the set of order preserving maps from L to M . It is itself a frame or locale under pointwise

inequality. L^{op} is the lattice whose members are the same as those of L but with reversed order, and

$$L \times M = Hom(L^{op}, M)$$

just as for vector spaces. The operations of a quantaloid make use of these operations on sup complete lattices.

The open subsets of a topological space form a frame or locale. The category of frames has for objects frames and for morphisms maps of lattices preserving \leq and satisfying the law

$$f(\vee a_i) \leq \vee f(a_i).$$

We call f a sup-lattice morphism

In general, sup lattice morphisms do **not** respect infinite infs.

The category of locales is the opposite category to the category of frames. In other words, a locale map is a frame map interpreted as going in the opposite direction.

The motivation for this is largely in the example of topological spaces. A continuous map takes open sets backwards into open sets, but not in general forwards. Thus, locales generalize topological spaces, and locale maps generalize continuous maps.

A locale is a special case of a quantale, in which \wedge and \circ coincide, \wedge distributes over \vee and \circ is commutative.

Locales can be treated very similarly to topological spaces, but they can be much more general. In fact, there exist pointless locales [2] .

It has proven very useful to think of a locale as a category whose objects are the elements of the locale, with one morphism from a to b if $a \leq b$ and none otherwise. This category is called the **site** of the locale.

This definition enables us to consider structures similar to the ones familiar in Physics, such as bundles, defined on any locale.

DEFINITION: *A presheaf on a locale is a contravariant functor from its site to the category of sets.*

This definition unpacks to a set for every element of the lattice with a restriction map from any element to any smaller element.

The physically minded reader might find it a useful exercise to check that if the locale is a manifold then the local sections of a bundle form a presheaf.

A presheaf which satisfies a certain glueing property is called a sheaf.

Now in ordinary set theoretic terms, a sheaf is a presheaf which satisfies the following simple condition:

DEFINITION: (glueing property) *if S is a presheaf over a space X such that if U_i is a cover of U and for all U_i in the cover $p(U_i) \in S(U_i)$ such that*

$$p(U_i) \mid_{U_i \cap U_j} = p(U_j) \mid_{U_i \cap U_j}$$

then there exists a unique $p(U) \in S(U)$

with

$$p(U) \mid_{U_i} = p(U_i) \text{ for all } i,$$

where \mid denotes the restriction map of the presheaf.

Functions and cross sections of bundles form sheaves.

Now in the general setting of locales or categories, it has turned out to be useful to define a presheaf as a contravariant functor, and a sheaf as a presheaf that satisfies the glueing property, but only for a restricted set of covers.

The sets of covers for which this definition of covers turns out to be useful are called **Grothendieck topologies**. We will not list their axioms here since they are somewhat technical, and since they are subsumed by quantalic nuclei on $\text{Rel}(\mathcal{C})$ as discussed in III.B. see [2].

DEFINITION: *a category with a Grothendieck topology is called a Grothendieck site. The category of sheaves over a Grothendieck site is called a Grothendieck topos.*

The category of sheaves over a locale with the Grothendieck topology of all open covers is called a **localic topos**. This is not the most general example of a topos, but it is not far from it, in that a deep theorem [3] tells us that any topos is the category of equivariant sheaves of some locale under the action of some semigroup.

The mathematical interest in topoi is largely due to the fact that they are so similar to the category of sets that it is possible to do all branches of Mathematics in a topos, where everything we know comes out different in various ways.

The real power of topos theory is the ability to choose between many different Grothendieck topologies. Since they are a special case of quantaloidal nuclei, the theory of sheaves over quantaloids is at least as powerful.

Let us describe an example which has interesting connections to quantum mechanics. Recall first the fact that a vector in the Hilbert space of a free particle is not a function, but rather an L^2 function, which is really an equivalence class of functions, and therefore does not have a value at any point. This suggests that ordinary quantum mechanics should admit a pointless formulation.

Now let us describe the Scott topos [5]. Let Λ be the lattice of Borel measurable subsets of some region R in R^3 , considered as a category with inclusions as morphisms (it is not quite a locale). Consider the Grothendieck topology of all covers of any measurable subset S of R whose union contains S up to a set of measure zero. The sheaves over this category with glueing over these covers form the Scott topos.

In this topos, which has no points, real numbers are exactly measurable functions. Ordinary quantum mechanics can easily be written in this topos. The integral is added in a straightforward way.

So we see how our new language allows us to express a certain filtering of information which is actually necessary in quantum physics but generally taken for granted. The intuition that exotic unmeasurable subsets of \mathbb{R} are unphysical can be put on a mathematical footing.

APPENDIX B. Distributive and non-distributive lattices in Physics

DEFINITION: *A world element is a proposition about a system which is either true or false.*

This definition, due originally to Einstein, was one of the important early motivations for lattice theory.

In classical mechanics, world elements correspond to Borel measurable subsets of phase space. In quantum mechanics, on the other hand, they are closed projections on Hilbert space.

There is an important difference between these two types of lattice: the first satisfies the distributive law:

$$a \wedge (\vee_{i \in I} b_i) = \vee_{i \in I} (a \wedge b_i)$$

while the second does not. This was the observation which motivated Van Neumann and Birkhoff to invent quantum logic.

The failure of the distributive law is a rather abstract way to describe many of the phenomena that make quantum mechanics so special and mysterious. For example, in the two slit experiment, the probability that the particle goes through both slits and then hits some spot is not the sum of the two probabilities of going through each slot then hitting the spot.

On the other hand, if we study only operations on a system in quantum mechanics which correspond to some subset of the operators which commute with one another, then we get a distributive lattice, and in fact the multiplication of the operators corresponds to the intersection of the world elements. (In the two slit experiment, the observation of the position of the spot happens at a later time than passage through the slits, and therefore does not commute with them).

In the quantale associated to a quantum mechanical system, sublocales correspond to commutative subalgebras of the C^* algebra.

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